Problem Set 7 – Relativity (G0I36A)

1 You're Killing Me, Schwarzschild

1.) Remember, Killing vectors correspond to isometries of the metric, i.e. coordinate changes that leave the metric invariant. Since all components of the Schwarzschild metric are independent of t, any time translation will leave the metric invariant. This corresponds to a Killing vector of the form

$$K = \partial_t \ . \tag{1.1}$$

For the remaining three Killing vectors, notice that the only angular dependence in the metric shows up in the $d\Omega^2$ term and nowhere else. This means that any isometry of S^2 will also automatically be an isometry of the Schwarzschild metric. There are three linearly independent Killing vectors on S^2; they correspond to rotations around the x, y, and z axes. These three Killing vectors are also Killing vectors of the full Schwarzschild metric. This gives us four Killing vectors overall.

2.) Any system that has time-translation invariance will have a conserved quantity that corresponds to the energy of the system. Thus, the Killing vector associated with time translations will lead to conservation of energy.

The three remaining Killing vectors, which generate rotations on S^2, correspond to conservation of angular momentum, which is a vector with three components.

2 A First Look at AdS$_2$

3.) The obvious first Killing vector, which we will denote by K_0, is

$$K_0 = \partial_\tau \ , \tag{2.1}$$

since all metric components are independent of τ. To find the remaining Killing vectors, we need to look at the Killing vector equation.

The non-zero Christoffel symbols are

$$\Gamma^\eta_{\tau\tau} = \cosh\eta \sinh\eta \ , \quad \Gamma^\tau_{\eta\tau} = \Gamma^\tau_{\tau\eta} = \tanh\eta \ . \tag{2.2}$$

The most general Killing vector on this background will take the form

$$K = f(\eta,\tau)\partial_\eta + g(\eta,\tau)\partial_\tau \ , \tag{2.3}$$

for some functions f and g. In component form, this means that

$$K_\eta = f(\eta,\tau) \ , \quad K_\tau = -\cosh^2\eta \, g(\eta,\tau) \ . \tag{2.4}$$

The Killing vector equation $\nabla_{(\mu}K_{\nu)} = \partial_{(\mu}K_{\nu)} - \Gamma^\rho_{\mu\nu}K_\rho = 0$ has three independent components:

$$\begin{aligned} (\eta,\eta): &\quad 0 = \partial_\eta K_\eta \ , \\ (\tau,\tau): &\quad 0 = \partial_\tau K_\tau - \Gamma^\eta_{\tau\tau} K_\eta \ , \\ (\eta,\tau): &\quad 0 = \partial_\tau K_\eta + \partial_\eta K_\tau - 2\Gamma^\tau_{\eta\tau} K_\tau \ . \end{aligned} \tag{2.5}$$

Writing these out explicitly and simplifying as much as possible, they give us the following three equations:

$$\begin{aligned} &(1) \quad 0 = \partial_\eta f(\eta,\tau) \ , \\ &(2) \quad 0 = \partial_\tau g(\eta,\tau) + \tanh\eta\, f(\eta,\tau) \ , \\ &(3) \quad 0 = \partial_\tau f(\eta,\tau) - \cosh^2\eta\, \partial_\eta g(\eta,\tau) \ . \end{aligned} \tag{2.6}$$

The first equation is easily solved; we simply have to set

$$f(\eta,\tau) = f(\tau) \ , \tag{2.7}$$

i.e. we have to demand that f depends only on τ, not η. Plugging this into the second equation then tells us that

$$\partial_\tau g(\eta,\tau) = -\tanh\eta\, f(\tau) \ , \tag{2.8}$$

which can be solved by the ansatz

$$g(\eta,\tau) = \tanh\eta\, h(\tau) \ , \quad h'(\tau) = -f(\tau) \ . \tag{2.9}$$

Plugging this into the third equation, we find that

$$0 = f'(\tau) - \cosh^2\eta\, \partial_\eta \left(\tanh\eta\, h(\tau)\right) = f'(\tau) - h(\tau) \ . \tag{2.10}$$

Then, since $h'(\tau) = -f(\tau)$, we also know that $h''(\tau) = -f'(\tau)$. The third and final equation thus boils down to the simple condition

$$h''(\tau) = -h(\tau) \ . \tag{2.11}$$

This is a second-order differential equation with two linearly independent solutions $h_1(\tau)$ and $h_2(\tau)$, which we can pick to be

$$h_1(\tau) = e^{i\tau} \ , \quad h_2(\tau) = e^{-i\tau} \ . \tag{2.12}$$

Plugging these back into the ansatz (2.9), we find that the corresponding solutions for f and g are given by:

$$f_1(\tau) = -ie^{i\tau} \ , \quad g_1(\eta,\tau) = \tanh\eta\, e^{i\tau} \ , \tag{2.13}$$

and

$$f_2(\tau) = ie^{-i\tau} \ , \quad g_2(\eta,\tau) = \tanh\eta\, e^{-i\tau} \ . \tag{2.14}$$

Therefore the two remaining Killing vectors can thus be taken as

$$\begin{aligned} K_1 &= e^{i\tau}\left(-i\partial_\eta + \tanh\eta\, \partial_\tau\right) \ , \\ K_2 &= e^{-i\tau}\left(i\partial_\eta + \tanh\eta\, \partial_\tau\right) \ . \end{aligned} \tag{2.15}$$

Now, if we compute the commutation relations between the Killing vectors explicitly, we find that

$$\begin{aligned} [K_0, K_1] &= iK_1 \ , \\ [K_0, K_2] &= -iK_2 \ , \\ [K_1, K_2] &= -2iK_0 \ . \end{aligned} \tag{2.16}$$

These look very similar to the $SL(2,\mathbb{R})$ commutation relations, except with some extra numerical factors off. If we identify

$$L_0 = -iK_0 \ , \quad L_+ = \frac{1}{\sqrt{2}}K_1 \ , \quad L_- = \frac{1}{\sqrt{2}}K_2 \ , \tag{2.17}$$

then the commutation relations are exactly the $SL(2,\mathbb{R})$ ones. Therefore the Killing vectors

$$\begin{aligned} L_0 &= -i\partial_\tau \ , \\ L_\pm &= \frac{1}{\sqrt{2}} e^{\pm i\tau} (\mp i\partial_\eta + \tanh\eta\,\partial_\tau) \ , \end{aligned} \tag{2.18}$$

of AdS_2 yield the isometry group $SL(2,\mathbb{R})$.

4.) **Bonus:** to compute the Ricci scalar for the spacetime, we first need to calculate the Christoffel symbols, which then let us compute the Riemann tensor, the Ricci tensor, and finally the Ricci scalar. As we established earlier in (2.2), the non-zero Christoffel symbols are given by

$$\Gamma^\eta_{\tau\tau} = \cosh\eta\,\sinh\eta \ , \quad \Gamma^\tau_{\eta\tau} = \Gamma^\tau_{\tau\eta} = \tanh\eta \ . \tag{2.19}$$

For the Riemann tensor, the symmetries of the Riemann tensor tell us that the only independent, non-trivial component is $R_{\eta\tau\eta\tau}$, which is easily written as

$$R_{\eta\tau\eta\tau} = g_{\eta\eta} R^\eta{}_{\tau\eta\tau} = R^\eta{}_{\tau\eta\tau} \ , \tag{2.20}$$

since the metric is diagonal and $g_{\eta\eta} = 1$. So, we need to compute $R^\eta{}_{\tau\eta\tau}$. The formula for the Riemann tensor in terms of Christoffel symbols tells us that

$$R^\eta{}_{\tau\eta\tau} = \partial_\eta \Gamma^\eta_{\tau\tau} - \partial_\tau \Gamma^\eta_{\eta\tau} + \Gamma^\eta_{\eta\lambda}\Gamma^\lambda_{\tau\tau} - \Gamma^\eta_{\tau\lambda}\Gamma^\lambda_{\eta\tau} \ . \tag{2.21}$$

The second term and the third term vanish, since we have established that $\Gamma^\eta_{\eta\mu} = 0$ for any μ. For the fourth term, only the $\lambda = \tau$ component will contribute in the sum over λ, and we are left with

$$\begin{aligned} R^\eta{}_{\tau\eta\tau} &= \partial_\eta \Gamma^\eta_{\tau\tau} - \Gamma^\eta_{\tau\tau}\Gamma^\tau_{\eta\tau} \\ &= \partial_\eta (\cosh\eta\,\sinh\eta) - \cosh\eta\,\sinh\eta\,\tanh\eta \\ &= \cosh^2\eta + \sinh^2\eta - \sinh^2\eta \\ &= \cosh^2\eta \ . \end{aligned} \tag{2.22}$$

Combining (2.20) and (2.22), we thus find that $R_{\eta\tau\eta\tau} = \cosh^2\eta$ is the only non-trivial independent component of the Riemann tensor. From here, we can contract over indices to compute the Ricci tensor components:

$$\begin{aligned} R_{\tau\tau} &= g^{\mu\nu} R_{\tau\mu\tau\nu} = g^{\eta\eta} R_{\tau\eta\tau\eta} = R_{\eta\tau\eta\tau} = \cosh^2\eta \ , \\ R_{\eta\eta} &= g^{\mu\nu} R_{\eta\mu\eta\nu} = g^{\tau\tau} R_{\eta\tau\eta\tau} = -\frac{1}{\cosh^2\eta} \times \cosh^2\eta = -1 \ , \\ R_{\eta\tau} &= g^{\mu\nu} R_{\eta\mu\tau\nu} = -g^{\mu\nu} R_{\eta\mu\nu\tau} = -g^{\tau\eta} R_{\eta\tau\eta\tau} = 0 \ . \end{aligned} \tag{2.23}$$

Note that the Ricci tensor is symmetric so $R_{\tau\eta} = R_{\eta\tau} = 0$. Finally, contracting over the indices of the Ricci tensor will give us the Ricci scalar, and since the off-diagonal pieces of the Ricci tensor vanish this contraction is easy:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\eta\eta} R_{\eta\eta} + g^{\tau\tau} R_{\tau\tau} = 1 \times -1 + \left(-\frac{1}{\cosh^2\eta}\right) \times \cosh^2\eta = -1 - 1 = -2 \ . \tag{2.24}$$

Therefore we have proven that $R = -2$ for our AdS_2 metric.

Now, for the analytic continuation part of the problem. First, let's consider what happens when we set $\theta = i\eta + \frac{\pi}{2}$. Since constant shifts don't matter for computing differentials, this means that $d\theta = i d\eta$, and thus $d\theta^2 = -d\eta^2$. Then, using a standard trig identity for shifts by $\pi/2$, we have

$$\sin\theta = \sin\left(i\eta + \frac{\pi}{2}\right) = \cos\left(i\eta\right) \ . \tag{2.25}$$

Importantly, hyperbolic trig functions are such that $\cos(i\eta) = \cosh\eta$, and thus we have that $\sin^2\theta = \cosh^2\eta$. Finally, setting $\phi = \tau$ doesn't do anything except change the coordinate range on the variable, and so $d\phi^2 = d\tau^2$. Putting this all together, the analytic continuation of the S^2 metric is therefore given by:

$$\begin{aligned} ds^2_{S^2} &= d\theta^2 + \sin^2\theta\, d\phi^2 \\ &\to \left(-d\eta^2\right) + \cosh^2\eta\, d\tau^2 \\ &= -\left(d\eta^2 - \cosh^2\eta\, d\tau^2\right) \\ &= -ds^2_{\mathrm{AdS}_2} \ . \end{aligned} \tag{2.26}$$

Therefore the analytic continuation proposed in the problem sends the line element on S^2 to the line element on AdS_2, with an overall minus sign. In fact, the relation between the two line elements is exactly what causes the Ricci scalar $R = 2$ on S^2 to get mapped to $R = -2$ on AdS_2, though the full details of this are beyond the scope of what we want to do in this problem. For now, just think of this as a neat coincidence.